lecture 7 – continuous time Markov models, and hidden Markov models

Continuous time Markov models (CTMC) may have discrete states, but instead of $x_1, x_2, x_3, \ldots$ we have a $x(t)$ which expresses the state at any value for $t > 0$.

If the rates of transition only depend on the current state, then it is still a Markov process. So, if we had 3 discrete states

$$P(t) = \begin{bmatrix} \mathbb{P}(X(t)=1 \mid x_0=1) & \mathbb{P}(X(t)=2 \mid x_0=1) & \mathbb{P}(X(t)=3 \mid x_0=1) \\ \mathbb{P}(X(t)=1 \mid x_0=2) & \mathbb{P}(X(t)=2 \mid x_0=2) & \mathbb{P}(X(t)=3 \mid x_0=2) \\ \mathbb{P}(X(t)=1 \mid x_0=3) & \mathbb{P}(X(t)=2 \mid x_0=3) & \mathbb{P}(X(t)=3 \mid x_0=3) \end{bmatrix} \tag{1}$$

Or more generally each element is $\mathbb{P}(X(t) = j \mid x_0 = i)$.

How can we calculate $P(t)$? It is the solution to a series of differential equations.

$$P(t) = e^{tQ} \tag{2}$$

where Q is a matrix of the instantaneous rates of change from each state to each other state.

e.g. for the chromosome example from last lecturea

$$Q = \begin{bmatrix} q_{AA\to AA} & q_{AB\to AA} & q_{BB\to AA} \\ q_{AA\to AB} & q_{AB\to AB} & q_{BB\to AB} \\ q_{AA\to BB} & q_{AB\to BB} & q_{BB\to BB} \end{bmatrix} \tag{3}$$

$$= \begin{bmatrix} -2r & r & 0 \\ 2r & -2r & 2r \\ 0 & r & -2r \end{bmatrix} \tag{4}$$

To solve for the stationary distribution:

$$Q\pi = 0 \tag{5}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2r & r & 0 \\ 2r & -2r & 2r \\ 0 & r & -2r \end{bmatrix} \begin{bmatrix} \pi_{AA} \\ \pi_{AB} \\ \pi_{BB} \end{bmatrix} \tag{6}$$

$$-2r\pi_{AA} + r\pi_{AB} + 0\pi_{BB} = 0 \tag{7}$$

$$\pi_{AB} = 2\pi_{AA} \tag{8}$$

If your data is in terms of waiting times until the next change of state, then we can model this as an exponential distribution with a hazard parameter that is the diagonal of the Q matrix.