notes from the week of March 06, 2019

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1 Birth-death Markov chain notes

The state space is the non-negative integers with the only possible changes in a single instant being an increase or decrease of 1.

If we had been working in discrete time, our difference equations would have been:

\begin{align*}
p_0(t) &= -\lambda_0 p_0(t - 1) + \mu_1 p_1(t - 1) \\
p_i(t) &= \lambda_{i-1} p_{i-1}(t - 1) - (\lambda_i + \mu_i) p_i(t - 1) + \mu_{i+1} p_{i+1}(t - 1)
\end{align*}

where \( \lambda_i \) represents the probability of a birth if you are in state \( i \) (an \( i \to i + 1 \) transition), and \( \mu_i \) the probability of a death if you are in state \( i \) (an \( i \to i - 1 \) transition). \( p_i(t) \) is the probability of being in state \( i \) at time (or iteration) \( t \).

But we were working in continuous time, so the general form of the differential equations is:

\begin{align*}
\frac{\partial p_0(t)}{\partial t} &= -\lambda_0 p_0(t) + \mu_1 p_1(t) \\
\frac{\partial p_i(t)}{\partial t} &= \lambda_{i-1} p_{i-1}(t) - (\lambda_i + \mu_i) p_i(t) + \mu_{i+1} p_{i+1}(t)
\end{align*}

where \( \lambda_i \) and \( \mu_i \) now represent rates of births and deaths. Because we are working in continuous time, we don’t have \( t - 1 \) representing the previous iteration, instead we make our fundamental statements based on the instantaneous rates of change.

Frequently we use \( \pi_i \) to represent the equilibrium probability of being in state \( i \). JKK just used \( p_i \) without the \( (t) \) after it.

If there is an equilibrium, it will be a dynamic one. We can start by assuming that one exists and seeing if we can solve for it. From equation 3 for state 0 we get:

\begin{align*}
0 &= -\lambda_0 \pi_0 + \mu_1 \pi_1 \\
\pi_0 &= \left( \frac{\mu_1}{\lambda_0} \right) \pi_1 \\
\pi_1 &= \left( \frac{\lambda_0}{\mu_1} \right) \pi_0
\end{align*}
We use equation 4 for state 1 to get:

\[ 0 = \lambda_i \pi_i - \pi_{i-1} - (\lambda_i + \mu_i) \pi_i + \mu_{i+1} \pi_{i+1} \]  
\[ 0 = \lambda_0 \pi_0 - (\lambda_1 + \mu_1) \pi_1 + \mu_2 \pi_2 \]  
\[ 0 = \mu_1 \pi_1 - (\lambda_1 + \mu_1) \pi_1 + \mu_2 \pi_2 \]  
\[ 0 = -\lambda_1 \pi_1 + \mu_2 \pi_2 \]  
\[ \pi_1 = \left( \frac{\mu_2}{\lambda_1} \right) \pi_2 \]  
\[ \pi_2 = \left( \frac{\lambda_1}{\mu_2} \right) \pi_1 = \left( \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} \right) \pi_0 \]  

We are building up higher values of \( i \) by balancing out the flow between the state \( i \) and \( i - 1 \). Equation (7) expresses the relationship for state 1 in terms of its \( i - 1 \) neighbor (state 0), so it may not be surprising that we continue this and Equation (13) for \( \pi_2 \) shows up as a balance between rates with states 1 and 2.

If you continue this process you get:

\[ \pi_i = \pi_0 \left( \frac{\lambda_0 \lambda_1 \ldots \lambda_{i-1}}{\mu_1 \mu_2 \ldots \mu_i} \right) \]  
\[ \pi_i = \pi_0 \prod_{j=1}^{i} \frac{\lambda_{j-1}}{\mu_j} \]  

which is pleasingly terse.

1.1 special case: state-independent transitions rates

If \( \lambda \) is the same for all \( i \) and \( \mu \) is a constant for all states then Equation (15), then it can be convenient to reparameterize into a ratio of \( \lambda \) and \( \mu \):

\[ r = \frac{\lambda}{\mu} \]  
\[ \pi_i = \pi_0 \prod_{j=1}^{i} \frac{\lambda_{j-1}}{\mu_j} \]  
\[ = \pi_0 \prod_{j=1}^{i} \frac{\lambda}{\mu} \]  
\[ = \pi_0 \prod_{j=1}^{i} r \]  
\[ \pi_i = \pi_0 r^i \]  

that is pleasingly even terser.

It may look like that is simple enough, but if you know \( \lambda \) and \( \mu \), you’d just have the tautology \( \pi_0 = \pi_0 \) if you plug in \( i = 0 \) into equation (20). We could look up the geometric distribution in Wikipedia, or we could use some probability theory:

\[ 1 = \sum_{i=0}^{\infty} \pi_i \]  

2
\[
\begin{align*}
\sum_{i=0}^{\infty} (\pi_0 r^i) &= \sum_{i=0}^{\infty} r^i & (22) \\
\pi_0 \sum_{i=0}^{\infty} r^i &= \pi_0 \sum_{i=0}^{\infty} r^i & (23)
\end{align*}
\]

because \(1 \times r = r\), we can get cute:

\[
\begin{align*}
\sum_{i=0}^{\infty} \pi_i &= \sum_{i=0}^{\infty} \pi_0 r^i & (24) \\
\pi_0 \sum_{i=0}^{\infty} (\pi_0 r^i) &= \pi_0 \sum_{i=0}^{\infty} r^{1+i} & (25) \\
\sum_{i=0}^{\infty} (\pi_0 r^{1+i}) &= \pi_0 \sum_{i=0}^{\infty} r^{1+i} & (26) \\
\sum_{i=0}^{\infty} (\pi_0 r^{1+i}) &= \pi_0 \sum_{i=0}^{\infty} r^{1+i} & (27)
\end{align*}
\]

this enables some even further cuteness where we play with the bounds of the sum:

\[
\begin{align*}
1 - r &= \pi_0 \left( \sum_{i=0}^{\infty} r^i \right) - \pi_0 \left( \sum_{i=0}^{\infty} r^{i+1} \right) & (28) \\
&= \pi_0 \left( \sum_{i=0}^{\infty} r^i - \sum_{i=1}^{\infty} r^i \right) & (29) \\
&= \pi_0 \left( r^0 + \sum_{i=1}^{\infty} r^i - \sum_{i=1}^{\infty} r^i \right) & (30) \\
&= \pi_0 & (31)
\end{align*}
\]

So:

\[\pi_i = (1 - r)r^i\]

is the general form of the equilibrium state frequency.

### 1.2 special case: each individual has the same birth and death rate

If \(\lambda_i = i\lambda\) and \(\mu_i = i\mu\) we run into problems with Equation (15) because the flux between state 0 and state 1 is never balanced (since 0 is an absorbing state).