$$\mathbb{P}(\alpha_0, \delta_1, \sigma_B, \sigma_C, \sigma_E | Y) = \frac{\mathbb{P}(Y | \alpha_0, \delta_1, \sigma_B, \sigma_C, \sigma_E)\mathbb{P}(\alpha_0, \delta_1, \sigma_B, \sigma_C, \sigma_E)}{\mathbb{P}(Y)}$$

$$\mathbb{P}(\alpha_0, \delta_1, \sigma_B, \sigma_C, \sigma_E) = \mathbb{P}(\alpha_0)\mathbb{P}(\delta_1)\mathbb{P}(\sigma_B)\mathbb{P}(\sigma_C)\mathbb{P}(\sigma_E)$$

If V is the variance-covariance matrix:

$$\mathbb{P}(Y | \alpha_0, \delta_1, \sigma_B, \sigma_C, \sigma_E) = K - \frac{\ln |V|}{2} - \frac{1}{2}(Y - X\theta)^T V^{-1}(Y - X\theta)$$

Note that V^{-1} is worse than an $\mathcal{O}(n^2)$ operation.

If we knew each family effect (B_j) and (C_j), things get easier.

Note that if we "integrate out" $\boldsymbol{B}$ and $\boldsymbol{C}$, we are still using the same model. The likelihood of our original parameters is produced by integration:

$$\mathbb{P}(Y|\alpha_0, \delta_1, \sigma_B, \sigma_C, \sigma_e) = \int \int \mathbb{P}(Y|\alpha_0, \delta_1, \boldsymbol{B}, \boldsymbol{C}, \sigma_e)\mathbb{P}(\boldsymbol{B}|\sigma_B)\mathbb{P}(\boldsymbol{C}|\sigma_C) d\boldsymbol{B} d\boldsymbol{C}$$

$$
\begin{aligned}
L(\alpha_0, \delta_1, B_j, C_j, \sigma_e) &= \frac{e^{\frac{-\left(y_{ijk}-\alpha_0-i\delta_1-B_j-iC_j\right)^2}{2\sigma_e^2}}}{\sqrt{2\pi\sigma_e^2}} \\
\ln L(\alpha_0, \delta_1, B_j, C_j, \sigma_e) &= K - \ln\sigma_e - \frac{\left(y_{ijk} - \alpha_0 - i\delta_1 - B_j - iC_j\right)^2}{2\sigma_e^2} \\
B_j &\sim \mathcal{N}(0, \sigma_B) \\
C_j &\sim \mathcal{N}(0, \sigma_C) \\
\mathbb{P}(\boldsymbol{Y}|\alpha_0, \delta_1, \boldsymbol{B}, \boldsymbol{C}, \sigma_e) &= \prod_{i=0}^{1}\prod_{j=1}^{n_f}\prod_{k=1}^{n_{0j}} \mathbb{P}(y_{ijk}|\alpha_0, \delta_1, B_j, C_j, \sigma_e)
\end{aligned}
$$

where n_f is the $\#$ families and n_0j is the number of individuals from family j that received treatment 0

The likelihood calculation is $\mathcal{O}(n)$, but the number of variables to update via MCMC is now $5 + 2n_f$. There will be more autocorrelation ("slower mixing").

We can get a big payoff, if there are tricks to calculating the likelihood ratio. Consider a $\alpha_0 \rightarrow \alpha_0^\star$ proposal:

$$\ln LR = \ln L(\alpha_0^\star, \delta_1, \boldsymbol{B}, \boldsymbol{C}, \sigma_e) - \ln L(\alpha_0, \delta_1, \boldsymbol{B}, \boldsymbol{C}, \sigma_e)$$

Separating by treatment:

$$\begin{aligned}
\ln LR_0 &= \sum_{j=1}^{n_f} \sum_{k=1}^{n_{0j}} \left[K - \ln \sigma_e - \frac{(y_{0jk} - \alpha_0^\star - B_j)^2}{2\sigma_e^2} - K + \ln \sigma_e + \frac{(y_{0jk} - \alpha_0 - B_j)^2}{2\sigma_e^2} \right] \\
&= \sum_{j=1}^{n_f} \sum_{k=1}^{n_{0j}} \left[\frac{1}{2\sigma_e^2} \left[(y_{0jk} - B_j - \alpha_0)^2 - \left(y_{0jk} - B_j - \alpha_0^\star \right)^2 \right] \right] \\
&= \frac{1}{2\sigma_e^2} \sum_{j=1}^{n_f} \sum_{k=1}^{n_{0j}} \left[2\left(y_{0jk} - B_j \right) \left(\alpha_0^\star - \alpha_0 \right) + \alpha_0^2 - (\alpha_0^\star)^2 \right] \\
&= \frac{1}{2\sigma_e^2} \left[2\left(y_{0\star\star} - B_{0\star} \right) \left(\alpha_0^\star - \alpha_0 \right) + n_0 \alpha_0^2 - n_0 (\alpha_0^\star)^2 \right]
\end{aligned}$$

where $y_{0\star\star}$, $B_{0\star}$, and n_0 are produced by summing over individuals in treatment 0. If you maintain these sums, the likelihood calculation times is constant wrt the # of individuals!

$$\text{Hastings ratio} = \frac{\mathbb{P}\left(\text{proposing } \theta^\star \rightarrow \theta\right)}{\mathbb{P}\left(\text{proposing } \theta \rightarrow \theta^\star\right)}$$

Peter Green's recipe:

1. draw a vector of k random variates, $\boldsymbol{u}$, from a probability distribution, $g(\boldsymbol{u})$
2. Use a deterministic function, h, to "map" the current parameters, $\boldsymbol{\theta}$, and $\boldsymbol{u}$ to a the proposed vector of parameters: $\boldsymbol{\theta}^\star = h(\boldsymbol{\theta}, \boldsymbol{u})$
3. Consider the reverse move, and express the random variates, $\boldsymbol{u}^\star$, required in the reverse the proposal.
4. Express functions for the elements of $\{\boldsymbol{\theta}^\star, \boldsymbol{u}^\star\}$ as a function of $\{\boldsymbol{\theta}, \boldsymbol{u}\}$
5. Calculate the determinant of the Jacobian matrix:

$$J = \begin{vmatrix} \partial\theta_1^\star/\partial\theta_1 & \partial\theta_2^\star/\partial\theta_1 & \ldots & \partial u_1^\star/\partial\theta_1 & \ldots & \partial u_k^\star/\partial\theta_1 \\ \partial\theta_1^\star/\partial\theta_2 & \partial\theta_2^\star/\partial\theta_2 & \ldots & \partial u_1^\star/\partial\theta_2 & \ldots & \partial u_k^\star/\partial\theta_2 \\ \vdots & \vdots & \ddots & & \ldots & \vdots \\ \partial\theta_1^\star/\partial u_k & \partial\theta_2^\star/\partial u_k & \ldots & \partial u_1^\star/\partial u_k & \ldots & \partial u_k^\star/\partial u_k \end{vmatrix}$$

6. Calculate:

$$\text{Hastings ratio} = \frac{g\left(\boldsymbol{u}^\star\right)}{g\left(\boldsymbol{u}\right)}|J|$$