notes from the week of March 06, 2019

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1 Birth-death Markov chain notes

The state space is the non-negative integers with the only possible changes in a single instant being an increase or decrease of 1.

If we had been working in discrete time, our difference equations would have been:

$$p_0(t) = -\lambda_0 p_0(t-1) + \mu_1 p_1(t-1) \quad (1)$$

$$p_i(t) = \lambda_{i-1} p_{i-1}(t-1) - (\lambda_i + \mu_i)\, p_i(t-1) + \mu_{i+1} p_{i+1}(t-1) \quad (2)$$

where λ_i is represents the probability of a birth if you are in state i (an $i \rightarrow i+1$ transition), and μ_i the probability of a death if you are in state i (an $i \rightarrow i-1$ transition). $p_i(t)$ is the probability of being in state i at time (or iteration) t.

But we were working in continuous time, so the general form of the differential equations is:

$$\frac{\partial p_0(t)}{\partial t} = -\lambda_0 p_0(t) + \mu_1 p_1(t) \quad (3)$$

$$\frac{\partial p_i(t)}{\partial t} = \lambda_{i-1} p_i(t) - (\lambda_i + \mu_i)\, p_i(t) + \mu_{i+1} p_{i+1}(t) \quad (4)$$

where λ_i and μ_i now represent rates of births and deaths. Because we are working in continuous time, we don't have $t-1$ representing the previous iteration, instead we make our fundamental statements based on the instantaneous rates of change.

Frequently we use π_i to represent the equilibrium probability of being in state i. JKK just used p_i without the (t) after it.

If there is an equlibrium, it will be a dynamic one. We can start by assuming that one exists and seeing if we can solve for it. From equation 3 for state 0 we get:

$$0 = -\lambda_0 \pi_0 + \mu_1 \pi_1 \quad (5)$$

$$\pi_0 = \left(\frac{\mu_1}{\lambda_0}\right) \pi_1 \quad (6)$$

$$\pi_1 = \left(\frac{\lambda_0}{\mu_1}\right) \pi_0 \quad (7)$$

We can use equation 4 for state 1 to get:

$$
\begin{aligned}
0 &= \lambda_{i-1}\pi_{i-1} - (\lambda_i + \mu_i)\,\pi_i + \mu_{i+1}\pi_{i+1} && (8)\\
0 &= \lambda_0\pi_0 - (\lambda_1 + \mu_1)\,\pi_1 + \mu_2\pi_2 && (9)\\
0 &= \mu_1\pi_1 - (\lambda_1 + \mu_1)\,\pi_1 + \mu_2\pi_2 && (10)\\
0 &= -\lambda_1\pi_1 + \mu_2\pi_2 && (11)\\
\pi_1 &= \left(\frac{\mu_2}{\lambda_1}\right)\pi_2 && (12)\\
\pi_2 &= \left(\frac{\lambda_1}{\mu_2}\right)\pi_1 = \left(\frac{\lambda_0\lambda_1}{\mu_1\mu_2}\right)\pi_0 && (13)
\end{aligned}
$$

We are building up higher values of i by balancing out the flow between the state i and $i-1$. Equation (7) expresses the relationship for state 1 in terms of its $i-1$ neighbor (state 0), so it may not be surprising that we continues this and Equation (13) for π_2 shows up as a balance between rates with states 1 and 2.

If you continue this process you get:

$$
\begin{aligned}
\pi_i &= \pi_0\left(\frac{\lambda_0\lambda_1\ldots\lambda_{i-1}}{\mu_1\mu_2\ldots\mu_i}\right) && (14)\\
\pi_i &= \pi_0\prod_{j=1}^{i}\frac{\lambda_{j-1}}{\mu_j} && (15)
\end{aligned}
$$

which is pleasingly terse.

1.1 special case: state-independent transitions rates

If λ is the same for all i and μ is a constant for all states then Equation (15), then it can be convenient to reparameterize into a ratio of λ and μ:

$$
\begin{aligned}
r &= \frac{\lambda}{\mu} && (16)\\
\pi_i &= \pi_0\prod_{j=1}^{i}\frac{\lambda_{j-1}}{\mu_j} && (17)\\
&= \pi_0\prod_{j=1}^{i}\frac{\lambda}{\mu} && (18)\\
&= \pi_0\prod_{j=1}^{i} r && (19)\\
\pi_i &= \pi_0 r^i && (20)
\end{aligned}
$$

that is pleasingly even terser.

It may look like that is simple enough, but if you know λ and μ, you'd just have the tautology $\pi_0 = \pi_0$ if you plug in $i = 0$ into equation (20). We could look up the geometric distribution in Wikipedia, or we could use some probability theory:

$$
1 = \sum_{i=0}^{\infty}\pi_i \qquad (21)
$$

$$
\begin{aligned}
&= \sum_{i=0}^{\infty} \left(\pi_0 r^i\right) && (22)\\
&= \pi_0 \sum_{i=0}^{\infty} r^i && (23)
\end{aligned}
$$

because $1 \times r = r$, we can get cute:

$$
\begin{aligned}
r &= r \sum_{i=0}^{\infty} \pi_i && (24)\\
&= r \sum_{i=0}^{\infty} \left(\pi_0 r^i\right) && (25)\\
&= \sum_{i=0}^{\infty} \left(\pi_0 r^{1+i}\right) && (26)\\
&= \pi_0 \sum_{i=0}^{\infty} r^{1+i} && (27)
\end{aligned}
$$

this enables some even further cuteness where we play with the bounds of the sum:

$$
\begin{aligned}
1 - r &= \pi_0 \left(\sum_{i=0}^{\infty} r^i\right) - \pi_0 \left(\sum_{i=0}^{\infty} r^{i+1}\right) && (28)\\
&= \pi_0 \left(\sum_{i=0}^{\infty} r^i - \sum_{i=1}^{\infty} r^i\right) && (29)\\
&= \pi_0 \left(r^0 + \sum_{i=1}^{\infty} r^i - \sum_{i=1}^{\infty} r^i\right) && (30)\\
&= \pi_0 && (31)
\end{aligned}
$$

So:

$$\pi_i = (1 - r) r^i$$

is the general form of the equilibrium state frequency.

1.2 special case: each individual has the same birth and death rate

If $\lambda_i = i\lambda$ and $\mu_i = i\mu$we run into problems with Equation (15) because the flux between state 0 and state 1 is never balanced (since 0 is an absorbing state).